Technical Comments

Comment on "Stiffness Matrices for Sector Elements"

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THE class of element lately called "isoparametric"[2] has in fact a long history. I. C. Taig first used them in 1958,[3] while the present author discovered them independently in 1961 and subsequently proved convergence. However, their popularity owes little to academic considerations; they are used because they are adaptable and easily programed.

The authors assert that they are expensive,[1] and one usually finds that those making this assertion are concerned to achieve convergence of the terms of the stiffness matrix. This is misguided but forgivable, because the contrary ideas have been mentioned only briefly[4,5] and no formal proof has yet been published. It is the purpose of this comment to rectify this. The proof suggested itself following the observation that using numerically integrated plate-bending triangles[6] the stresses and displacements were sensitive to the order of integration formula used only when the mesh was anyway too coarse to give useful answers. In this case the integrands were such that accurate numerical integration was not possible using standard formulae. Evidently the behavior of an approximately integrated element better resembled that of the accurately integrated element the nearer one approached constant stress conditions over the element.

Indeed, a certain order of integration accuracy guarantees perfect response in a constant stress field. For a general isoparametric element, a formula that guarantees correct volume in general guarantees correct response under constant stress. It thus guarantees convergence because under fine-mesh conditions such approximately integrated elements cannot be distinguished from the correctly integrated elements, which if they allow constant strain and guarantee conformity[4] are known to converge.

The algebra is presented for a case of plane stress, with constant thickness "t" and with the scalar displacement functions $N_i(\xi,\eta)$ assumed to be polynomials.[7] The volume is found from the Jacobian $[J]$

$$t\iint \det\ [J]d\xi d\eta$$

where

$$\det\ [J] = (\Sigma x_i \partial N_i/\partial\xi)(\Sigma y_j \partial N_j/\partial\eta) - (\Sigma y_i \partial N_i/\partial\xi)(\Sigma x_j \partial N_j/\partial\eta) \quad (1)$$

We now consider the nodal forces corresponding to a constant value σ_x over the element, and the proof resembles that of the Addendum to Ref. 6. To discover the nodal force corresponding to displacement u_i, for example, we impose a virtual change δu_i and find the virtual work done against the stress σ_x:

$$\text{force } U_i = \sigma_x t \iint \partial N_i/\partial x \det\ [J]d\xi d\eta \quad (2)$$

But

$$\begin{Bmatrix} \partial N_i/\partial x \\ \partial N_i/\partial y \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \partial N_i/\partial \xi \\ \partial N_i/\partial \eta \end{Bmatrix}$$

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where

$$[J]^{-1} = \frac{1}{\det\ [J]} \begin{bmatrix} \Sigma y_j \partial N_j/\partial\eta, & -\Sigma y_j \partial N_j/\partial\xi \\ -\Sigma x_i \partial N_i/\partial\eta, & \Sigma x_j \partial N_j/\partial\xi \end{bmatrix}$$

Expanding gives

$$\text{force } U_i = \sigma_x t \iint \partial N_i/\partial\xi \Sigma y_j \partial N_j/\partial\eta - \partial N_i/\partial\eta \Sigma y_j \partial N_j/\partial\xi\, d\xi d\eta \quad (3)$$

Now comparing (1) and (3) reveals that if i and $j = 1$ to n, both are composed of the same n^2 polynomial terms. If these are of such degree that the given integration formula integrates them exactly over the element, both the volume and the nodal forces are accurately integrated.

This theorem has clear financial implications, because engineers demand some justification for using the cheaper integration rules. Such justification may be numerical. For example, one may take an element or an assembly of elements and impose boundary conditions consistent with a state of constant stress, and check for correct response. Alternatively two formulations of an element with the same geometry can be compared by finding their relative eigenvalues.[8] Given the two stiffness matrices K_1 and K_2, of which one or both may be approximately integrated, or formulated on a hybrid basis following Pian, etc. we find the λ for which $K_1\delta = \lambda K_2\delta$. The λ are 0/0 (indeterminate) for common rigid-body motions, occasionally 0/1 or 1/0 if one of the elements has spurious mechanisms—as can happen with very low-order integration rules—and it is precisely 1 for any analytically correct response, e.g., constant stress conditions, that the two elements have in common. Further, if the determinate λ are all greater than 1, element 1 is always stiffer than element 2. If both are accurately integrated elements with assumed displacements, this gives a decisive verdict; formulation 2 is better than formulation 1.

An approximately integrated element usually behaves in practice as if the stiffness were reduced, although the values of λ show that the strain energy is not always reduced. The bound theorems are lost unless quite high-order integration is used, whose cost is not justified.

A remarkable but not particularly useful corollary arose in Ref. 9. Here the nonconforming plate bending triangle due to Zienkiewicz, known to converge for particular mesh patterns (see Ref. 6, Addendum) was shown to converge even if it was approximately integrated.

A membrane shell element has been published[10] but it has not been extensively used because applications seldom arise in civil engineering. Those that do arise often give ruled surfaces, and with high-order elements Sohrab Ahmad found that these tended to produce embarrassing mechanisms.

References

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[5] Irons, B., "Numerical Integration Applied to Finite Element Methods," International Symposium on the Use of Elec-

trical Digital Computers in Structural Engineering, Univ. of Newcastle upon Tyne, 1966.

[6] Bazeley, G. P. et al, "Triangular Elements in Plate Bending—Conforming and Nonconforming Solutions," Conference on Matrix Methods, Air Force Institute of Technology, Dayton, Ohio, 1965; also see Addendum.

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[10] Ahmad, S., "Pseudo-Isoparametric Finite Elements for Shell and Plate Analysis," Joint British Committee on Stress Analysis Conference on Recent Advances in Stress Analysis, R.Ae.S., London, March 1968.

Reply by Authors to B. M. Irons

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THE authors are pleased that their Note should have led B. Irons to discuss an important aspect of convergence in the finite element methods. They are, in fact, concerned that the comments made may not be lost, being only part of a discussion, instead of being put out as a Note in its own right.

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However, the authors are not clear exactly where they have been "misguided," for they cannot trace any real contradiction between their statements and Dr. Irons' comments. The authors' viewpoint was that, whereas general polygonal elements with generalized functional descriptions are very useful for arbitrary shapes, for special shapes, like circular holes that are often used in design, it is relatively economical and convenient to apply simple tailored elements. They have the feeling that Dr. Irons is not really questioning this viewpoint, but is discussing a more general question of criteria for convergence with arbitrary elements and numerical integrations for elements.

On the other hand, the authors would like to take this opportunity to make good an oversight in their original Note. The trigonometric distributions of displacements, unlike the linear distributions (Table 1), do not adequately provide the necessary kinematic freedoms, so that one may run into convergence troubles when applying them.

Erratum: "Experimental Study of High-Enthalpy Shock-Tunnel Flow. Part I: Shock-Tube Flow and Nozzle Starting Time"

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THE correct title of Fig. 2 is "Incident-shock test time in air for laminar boundary layer."